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THE STABILITY OF SYSTEMS WITH A DELAY AT POINTS ON THE BOUNDARIES OF STABILITY DOMAINS WHERE SAFE SECTIONS BECOME UNSAFE ONES†

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The results obtained in [1–5] are applied to give a criterion for the stability of the equilibrium states of systems with a delay at points on the boundaries of stability domains where safe sections become unsafe.

The problem of determining unsafe and safe boundaries of stability domains for the equilibrium states of systems with a delay has been considered in [1–10]. Methods and algorithms for investigating the stability of systems with a delay in critical cases where they can be reduced to truncated systems without a delay are given in [6–10]. Formulae for a quantity similar to the first Lyapunov value have been obtained for first-order equations with a delay in [1, 5, 9]. However, no such simple and convenient criteria are given in [1–10] for the stability of the equilibrium states of systems with a delay at points of the boundaries of stability domains where safe sections become unsafe, such as exist for systems without a delay [11].

For systems described by the second-order scalar equation with delay

$$\ddot{x} = a_1 x + a_2 \dot{x} + b_1 x(t-\tau) + b_2 \dot{x}(t-\tau) + f(x, \dot{x}, x(t-\tau), \dot{x}(t-\tau)) \quad (1)$$

we will consider the problem of determining the stability at points of the boundaries of stability domains for the equilibrium states where safe sections become unsafe.

Suppose the analytic function $f(x_1, x_2, x_3, x_4)$ can be expanded in series in the neighbourhood of $x_1 = x_2 = x_3 = x_4 = 0$, which starts with terms of not lower than the second degree in $(x_1 = x_2 = x_3 = x_4)$ of the form

$$f = \sum_{1 \le i \le k \le 4} a_{ik} x_i x_k + \sum_{1 \le i \le k \le p \le 4} a_{ikp} x_i x_k x_p + \sum_{1 \le i \le k \le p \le s \le 4} a_{ikps} x_i x_k x_p x_s +$$
$$+ \sum_{1 \le i \le k \le p \le s \le m \le 4} a_{ikpsm} x_i x_k x_p x_s x_m + \ldots$$

where the coefficients a_{ik}, a_{ikp}, a_{ikps}, a_{ikpsm} are constant.

Suppose the characteristic equation

$$\Delta(p) = \begin{vmatrix} p & -1 \\ -a_1 - b_1 e^{-\tau p} & p - a_2 - b_2 e^{-p\tau} \end{vmatrix} = 0 \quad (2)$$

has simple roots $p_{1,2} = \pm i\omega$ and roots $p_j (j \ge 3)$ satisfying the condition $\operatorname{Re} p_j < -\sigma < 0$. In this case the stability of the equilibrium state $x = 0$ of Eq. (1) is determined by the sign of quantities similar to Lyapunov quantities [6–10].

Suppose the quantity similar to the first Lyapunov quantity for Eq. (1) is equal to zero, while the quantity similar to the second is non-zero.

With these assumptions, we shall investigate the stability of the state of equilibrium $x = 0$ of Eq. (1) by calculating the quantity similar to the second Lyapunov quantity and determining its sign.

We write Eq. (1) in the form

$$\dot{x}^* = Ax^* + Bx^*(t-\tau) + F(x^*, \ x^*(t-\tau)) \quad (3)$$

where the vector x^* has components $x^*_1 = x, x^*_2, \dot{x}$, the matrices $A = [a^*_{ik}]$, $B = [b^*_{ik}]$ $(i, k = 1, 2)$ have elements

$$a^*_{11} = b^*_{11} = b^*_{12} = 0, \ a^*_{12} = 1, \ a^*_{21} = a_1, \ a^*_{22} = a_2, \ b^*_{21} = b_1$$

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$b^*_{22} = b_2$ and the vector function $F(x^*, x^*(t-\tau))$ has components

$$F_1 = 0, \quad F_2 = f(x_1^*, x_2^*, x_1^*(t-\tau), x_2^*(t-\tau)).$$

Second-order systems with a delay were considered in [5]. Using the results of [5, 7], we write system (3) in operator form

$$dx_t(\theta)/dt = Lx_t(\theta) + R(x_t(\theta)) \tag{4}$$

$$Lx_t(\theta) = \begin{cases} dx_t(\theta)/dt, & -\tau \leqslant \theta < 0 \\ Ax_t + Bx_t(-\tau), & \theta = 0 \end{cases}$$

$$Rx_t(\theta) = \begin{cases} 0, & -\tau \leqslant \theta < 0 \\ F(x_t(0),\ x_t(-\tau)), & \theta = 0 \end{cases}$$

where $x_t(\theta) = x^*(t+\theta)$ and $x^*(t)$ is a vector with components $x_1^*(t), x_2^*(t)$ which is the solution of system (3) for $t > 0$ with continuously differentiable initial vector function

$$x_0(\theta) = \varphi(\theta), \quad \theta \in [-\tau, 0].$$

Let $\Delta_{ik}(p_j)$ be the cofactors of the elements of the ith row and kth column of the determinants $\Delta(p_j)$.

Consider the vector functions $\beta_j(\theta)$ with components

$$\beta_j^{(k)}(\theta) = \exp(p_j\theta)\Delta_{2k}(p_j)/\Delta_j; \quad k = 1,2; \; j = 1,2$$

and the values

$$\Delta_j = d\Delta(p)/dp\,|_{p=p_j} = 2p_j + e^{-p_j\tau}(\tau b_1 + \tau p_j b_2 - b_2) - a_2$$

Consider the functionals

$$f_j[x(\theta)] = \sum_{i=1}^{2} \Delta_{i1}(p_j)\left[x_i(0) + \sum_{l=1}^{2} \int_{-\tau}^{0} e^{-p_j(\nu+\tau)} x_l(\nu) b_{il}\, d\nu\right]$$

where $t_1(\theta), x_2(\theta)$ are the components of the vector $x(\theta)$.

In system (4) we replace the variables $x_t(\theta)$ by the variables $y_1, y_2, z_t(\theta)$ using the formulae

$$y_j(t) = f_j[x_t(\theta)], \quad z(\theta) = x(\theta) - \sum_{j=1}^{2} \beta_j(\theta) y_j(t)$$

Following the procedure in [7] we change from the system in new variables to a truncated second-order system without delay. We do this by replacing the variable $z_t(\theta)$ by the variable $R_t(\theta)$ according to the formula

$$z_t(\theta) = R_t(\theta) + \gamma(\theta, y_1, y_2) = R_t(\theta) + \sum_{k=2}^{4} \sum_{r+q=k} d_{rq} y_1^r y_2^q$$

where γ is a two-dimensional vector function.

The coefficients $d_{rq}(\theta)$, which are two-dimensional vector functions, are found from the operator equation

$$[J\lambda - L]d_{rq}(\theta) = B_{rq}(\theta) \tag{5}$$

where J is the identity operator, $B_{rq}(\theta)$ is a known function and $\lambda = (r-q)i\omega$.

Since if $r - q \neq \pm 1$ the quantity λ does not occur in the spectrum of the operator L, we obtain from (5)

$$d_{rq}(\theta) = R(\lambda, J)B_{rq}(\theta) \tag{6}$$

where $R(\lambda, J)$ is the resolvent of the operator $\{J\lambda - L\}$. From Eq. (6) for, $r - q \neq \pm 1$ we obtain

$$d_{rq}(0) = \chi^{-1}(\lambda)\left(D_{rq} - \sum_{j=1}^{2} A_{rq}\alpha_j + BC_{rq}\right)$$

$$d_{rq}(-\tau) = e^{-\lambda\tau} d_{rq}(0) + C_{rq} \tag{7}$$

$$C_{rq}=A_{rq}\sum_{j=1}^{2}\frac{1}{p_j-\lambda}\alpha_j(e^{-p_j\tau}-e^{-\lambda\tau})$$

$$\chi(\lambda)=(\lambda E-A-Be^{-\lambda\tau})$$

The two-dimensional vector D_{rq} has components $D_{rq}^{(1)}=0$ and $D_{rq}^{(2)}=A_{rq}$, the quantities A_{rq} will be given below and the vector α_j has components $\alpha_{1j}=1/\Delta_j$, $\alpha_{2j}=p_j/\Delta_j$ $(j=1,2)$, $\lambda=(r-q)i\omega$.

When $r-q\neq\pm1$, the coefficients cannot be found in the form (6), but can also be obtained from Eq. (5) [7].

When $r=2, q=1, r-q=1$ we can apply to (5) the technique explained in [7] to obtain the vector $d_{21}(\theta)$ with $\theta=0, \theta=-\tau$

$$d_{21}(0)=V-A_{21}\left[(\alpha_{11}+\alpha_{12})(1+\tau e^{-p_1\tau}b_2)-\frac{\tau^2}{2}e^{-p_1\tau}(b_1\alpha_{11}+b_2\alpha_{21})+\frac{1}{2p_1}(\Delta_{11}\alpha_{12}+\alpha_{22})+\right.$$

$$\left.+\frac{1}{2p_1}e^{-p_1\tau}\tau(b_1\alpha_{12}+b_2\alpha_{22})\right]\alpha_1 \tag{8}$$

$$d_{21}(-\tau)=e^{-p_1\tau}d_{21}(0)-A_{21}e^{-p_1\tau}\left[\tau\alpha_1+\frac{1}{2p_1}(e^{2p_1\tau}-1)\alpha_2\right]$$

$$\Delta_{11}=p_1-a_2-b_2e^{-p_1\tau}$$

The vector V has components $V^{(1)}=0$, $V^{(2)}=A_{21}(\alpha_{11}+\alpha_{12})$, the value of A_{21} will be given below.

The vectors $d_{12}(0)$ and $d_{12}(-\tau)$ are complex conjugates [7] of the vectors $d_{21}(0)$ and $d_{21}(-\tau)$, and can therefore also be found from formulae (8).

We shall use the following notation

$$d_{rq}^{(i)}=d_{rq}^{(i)}(0),\quad d_{rq}^{(i+2)}=d_{rq}^{(i)}(-\tau);\quad i=1,2;\quad 2\leqslant r+q\leqslant 4$$

The truncated second-order system without delay has the form

$$\dot{y}_j=p_jy_j+Q(y_1,y_2)=p_jy_j+\sum_{k\geqslant2}\sum_{r+q=k}A_{rq}y_1^ry_2^q,\quad j=1,2 \tag{9}$$

where

$$Q=f(\psi_1,\psi_2,\psi_3,\psi_4)=\sum_{1\leqslant i\leqslant k\leqslant4}a_{ik}\psi_i\psi_k+\sum_{1\leqslant i\leqslant k\leqslant p\leqslant4}a_{ikp}\psi_i\psi_k\psi_p+$$

$$+\sum_{1\leqslant i\leqslant k\leqslant p\leqslant s\leqslant4}a_{ikps}\psi_i\psi_k\psi_p\psi_s+\sum_{1\leqslant i\leqslant k\leqslant p\leqslant s\leqslant m\leqslant4}a_{ikpsm}\psi_i\psi_k\psi_p\psi_s\psi_m+\dots$$

$$\psi_j=\alpha_{j1}y_1+\alpha_{j2}y_2+\gamma^{(j)}(0,y_1,y_2),\ j=1,2$$

$$\psi_{j+2}=\alpha_{j+2,1}y_1+\alpha_{j+2,2}y_2+\gamma^{(j)}(-\tau,y_1,y_2)$$

$$\alpha_{kj}=e^{-p_j\tau}\alpha_{k-2,j},\quad k=3,4$$

$$\gamma^{(j)}(0,y_1,y_2)=\sum_{2\leqslant r+q\leqslant5}d_{rq}^{(j)}y_1^ry_2^q,\quad \gamma^{(j)}(-\tau,y_1,y_2)=\sum_{2\leqslant r+q\leqslant5}d_{rq}^{(j+2)}y_1^ry_2^q$$

A_{rq} are the constant coefficients which appear in formulae (7) and (8), given by the formula

$$A_{rq}=\frac{1}{r!q!}\frac{\partial^{r+q}Q(y_1,y_2)}{\partial y_1^r\partial y_2^q}\Big|_{y_1=y_2=0} \tag{10}$$

In particular, we have

$$A_{20}=\sum_{1\leqslant i\leqslant k\leqslant4}a_{ik}\alpha_{i1}\alpha_{k1},\quad A_{11}=\sum_{1\leqslant i\leqslant k\leqslant4}a_{ik}(\alpha_{i1}\alpha_{k2}+\alpha_{i2}\alpha_{k1})$$

$$A_{30}=\sum_{1\leqslant i\leqslant k\leqslant4}a_{ik}(\alpha_{i1}d_{20}^{(k)}+\alpha_{k1}d_{20}^{(i)})+\sum_{1\leqslant i\leqslant k\leqslant p\leqslant4}a_{ikp}\alpha_{i1}\alpha_{k1}\alpha_{p1}$$

$$A_{21}=\sum_{1\leqslant i\leqslant k\leqslant4}a_{ik}(\alpha_{i2}d_{20}^{(k)}+\alpha_{k2}d_{20}^{(i)}+\alpha_{k1}d_{11}^{(i)}+\alpha_{i1}d_{11}^{(k)})+$$

$$+ \sum_{1 \le i \le k \le p \le 4} a_{ikp}[\alpha_{p1}(\alpha_{i1}\alpha_{k2} + \alpha_{i2}\alpha_{k1}) + \alpha_{i1}\alpha_{k1}\alpha_{p2}] \quad (11)$$

The coefficients A_{02}, A_{03}, A_{12} and A_{20}, A_{30}, A_{21} are complex conjugates. When $r + q = 2$, the quantities $d_{rq}^{(k)}$ are found from formulae (7) using the expressions for A_{20}, A_{11}, A_{02}. When $r + q = 3$ the quantities $d_{rq}^{(i)}$ are found from the coefficients $A_{rq}(r + q = 3)$ of formulae (7) and (8). When $r + q = 4$ the coefficients A_{rq} are found using the quantities $d_{rq}^{(i)}(r + q = 3)$ from formula (1) and are then used to find $d_{rq}^{(i)}(r + q = 4)$ and the coefficients A_{rq} for $r + q = 5$.

The first Lyapunov quantity for system (9) is found [11, 12] from the formula

$$g_1 = \operatorname{Re} A_{21} - \omega^{-1} A_{11} \operatorname{Im}(A_{20})$$

The quantity g_1 for Eq. (1) is a quantity similar to the first Lyapunov quantity [7].

If $g_1 < 0$, the boundary of the stability domain for Eq. (1) is safe; if $g_1 > 0$, it is unsafe. Correspondingly, if $g_1 = 0$, as assumed here, the stability of the equilibrium state $x = 0$ of Eq. (1) is found from the sign of the second Lyapunov quantity of system (9), and is simultaneously [7] a quantity similar to the second Lyapunov quantity for Eq. (1).

A formula was obtained in [9] for the second Lyapunov quantity of the equation

$$\dot{z} = i\omega z + \sum_{k+j \ge 2} \frac{g_{kj}}{k!j!} z^k \bar{z}^j \quad (12)$$

in which the bar denotes the complex conjugate and g_{kj} are constant coefficients. The first equation of system (9) is the same as (12), and the coefficients A_{kj} are the same as $g_{kj}/(k!j!)$. Thus, the formula obtained for the second Lyapunov quantity in [9] can also be used for system (9).

If $g_1 = 0$ and $g_2 < 0$, the equilibrium state of a system with delay described by Eq. (1) is stable, and if $g_2 > 0$ it is unstable.

If the function f contains no quadratic terms, that is, if all $a_{ik} = 0$, the expression for g_2 is much simpler.

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REFERENCES

1. FISHMAN, L. Z., On the generation of a periodic solution in systems of differential equations with a delay. *Izv. Vuzov. Matematika*, 1976, 12, 96–107.
2. FISHMAN, L. Z., A criterion for determining unsafe and safe boundaries of the stability domains for systems with a delay. *Avtomatika i Telemekhanika*, 1987, 10, 185–187.
3. FISHMAN, L. Z., Criteria for unsafe and safe boundaries of the stability domains for systems with a delay in the case of a zero root. *Differents. Uravneniya*, 1990, **26**, 10, 1830–1832.
4. FISHMAN, L. Z., Criteria of unsafe and safe boundaries for equations with a delay. In *Nonlinear Oscillations of Mechanical Systems. Reports to 3rd Conference*. Nizhnii Novgorod, 1993, 190.
5. FISHMAN, L. Z., Criteria of unsafe and safe boundaries of the stability domain of equations with a delay. *Prikl. Mat. Mekh.*, 1997, **61**, 4, 706–713.
6. MYSHKIS, A. D., SHIMANOV, S. N. and EL'SGOL'D, L. Z., The stability and oscillations of systems with a delay. In *Proc. of the International Symposium on Nonlinear Oscillations*. Kiev, 1961, 2, 241–267. Izd. Akad. Nauk UkrSSR, Kiev, 1963.
7. SHIMANOV, S. N., A critical case of a pair of pure imaginary roots for systems with a delay. *Sib. Mat. Zh.* 1961, **2**, 3, 467–480.
8. KOLESOV, Yu. S. and SHVITRA, D. I., *Self-excited Oscillations in Systems with a Delay*. Mokslas, Vilnius, 1979.
9. HASSARD, B. D., KARARINOFF, N. D. and WAN, Y.-H., *Theory and Applications of Hopf Birfurcation*. Cambridge University Press, Cambridge, 1981.
10. HALE, J. K., *Functional Equations*. Appl. Math. Ser. Vol. 3. Springer, New York, 1971.
11. BAUTIN, N. N., *The Behaviour of Dynamical Systems near Boundaries of the Stability Domain*. Nauka, Moscow, 1984.
12. NEIMARK, Yu. I., *Dynamical Systems and Control Processes*. Nauka, Moscow, 1978.

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